Mathematics IV Frameworks Student Edition

Unit 3
Rational Functions

1st Edition
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Georgia Department of Education

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Mathematics III
Unit 3
Rational Functions
Student Edition

INTRODUCTION:

In Mathematics I, students first investigated the properties of simple rational functions. The purpose of this unit is to expand their knowledge of the graphical behavior and characteristics of more complex rational functions. Students will be required to recall and make use of their knowledge of polynomial functions as well as compositions of functions to investigate the characteristics of these more complex rational functions. In addition, students will make use of the properties of inverse functions in order to further analyze each rational function. Finally, applications of these rational functions will be introduced as well as an emphasis on interpretation of real world phenomena as it relates to certain characteristics of the rational expressions.

ENDURING UNDERSTANDINGS:

- Solve rational functions
- Interpret graphs and discover characteristics of rational functions
- Recognize rational functions a the division of two polynomial functions

KEY STANDARDS ADDRESSED:

MM4A1. Students will explore rational functions.

a. Investigate and explain characteristics of rational functions, including domain, range, zeros, points of discontinuity, intervals of increase and decrease, rates of change, local and absolute extrema, symmetry, asymptotes, and end behavior.
b. Find inverses of rational functions, discussing domain and range, symmetry, and function composition.
c. Solve rational equations and inequalities analytically, graphically, and by using appropriate technology.

RELATED STANDARDS ADDRESSED:

MM4A4. Students will investigate functions.

a. Compare and contrast properties of functions within and across the following types: linear, quadratic, polynomial, power, rational, exponential, logarithmic, trigonometric, and piecewise.
b. Investigate transformations of functions.
c. Investigate characteristics of functions built through sum, difference, product, quotient, and composition.

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MM4P1. Students will solve problems (using appropriate technology).

a. Build new mathematical knowledge through problem solving.
b. Solve problems that arise in mathematics and in other contexts.
c. Apply and adapt a variety of appropriate strategies to solve problems.
d. Monitor and reflect on the process of mathematical problem solving.

MM4P2. Students will reason and evaluate mathematical arguments.

a. Recognize reasoning and proof as fundamental aspects of mathematics.
b. Make and investigate mathematical conjectures.
c. Develop and evaluate mathematical arguments and proofs.
d. Select and use various types of reasoning and methods of proof.

MM4P3. Students will communicate mathematically.

a. Organize and consolidate their mathematical thinking through communication.
b. Communicate their mathematical thinking coherently and clearly to peers, teachers, and others.
c. Analyze and evaluate the mathematical thinking and strategies of others.
d. Use the language of mathematics to express mathematical ideas precisely.

MM4P4. Students will make connections among mathematical ideas and to other disciplines.

a. Recognize and use connections among mathematical ideas.
b. Understand how mathematical ideas interconnect and build on one another to produce a coherent whole.
c. Recognize and apply mathematics in contexts outside of mathematics.

MM4P5. Students will represent mathematics in multiple ways.

a. Create and use representations to organize, record, and communicate mathematical ideas.
b. Select, apply, and translate among mathematical representations to solve problems.
c. Use representations to model and interpret physical, social, and mathematical phenomena.

Unit Overview:

In the first task of this unit, students will remember the properties of polynomial functions and will investigate how the division of two of those polynomials will affect the characteristics of the resultant rational function. By the end of the task, students should be able to write explanations of how and why the rational function has certain characteristics based upon its polynomial pieces. The second task takes this same approach to helping students deduce the graphs of rational functions without the aid of a calculator. By using sign charts and critical points, students can begin to understand how and why a rational function must behave in the way that it does based on its characteristics examined in the first task. The third task takes a look at horizontal asymptotes and could be done before the second task (at the discretion of the teacher). By using tables, students will investigate function behavior as the input values increase towards positive and negative infinity and draw conclusions as to how the degree of the numerator and denominator affects the horizontal asymptote of the rational function. The fourth task lets students use their

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knowledge of inverse functions to investigate the inverses of certain rational functions. As not all rational functions are one to one, the inverse of a rational function may not always be useful in determining properties of the original. The fifth task asks students to discover how the summation of simple rational functions can form a more complex rational function, or partial fraction decomposition. In this task, students will learn a method of partial fraction decomposition and identify some uses of this technique in graphing rational functions. Finally, in the culminating task, students will analyze an extremely complex rational function for determining the efficiency rating of an NFL quarterback. By holding some statistics stable, students will create a rational function with two variables which can then be analyzed in conjunction with the properties of rational functions. Students will use their knowledge of the characteristics of rational functions to provide real world interpretations in this problem scenario. An alternate culminating task is also included, involving the appropriate dimensions to minimize the cost of a cylindrical coffee can. Students will have to solve for the appropriate dimensions, investigate the sizes of actual products, and compare the actual dimensions with the computed dimensions in order to determine why a product has such proportions.

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Rational Functions Characteristics Learning Task:

What do you know about the polynomial $g(x) = x^2 + 3x - 10$?

What is the Domain? How do you determine the Domain?

What is the Range? How do you determine the Range?

Where are the Roots or Zeros found? What are some different ways you know to find them?

What is the End Behavior? How do you know?

What do you know about the polynomial $f(x) = x + 1$?

What is the Domain?

What is the Range?

What are the Roots or Zeros?

What is the End Behavior? How do you know?

Now let's consider the case of the rational function $r(x) = \dfrac{f(x)}{g(x)}$ where f and g are the polynomial functions above.

What is the domain of $r(x)$? Which function, f or g, affects the domain the most? Why?

What do you think the range of $r(x)$ will be? Why is this so difficult to determine?

What are the roots or zeros of $r(x)$? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will $r(x)$ intersect the y-axis? How do you know?

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Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior?

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

Based on the graph from your calculator, what is the range of $r(x)$?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Let's try a few more problems and see if we can discover any patterns…

1. Let $f(x) = 5$ and $g(x) = x^2 - 6x + 8$. Let $r(x) = \dfrac{f(x)}{g(x)}$.

What is the domain of $r(x)$? Which function, f or g, affects the domain the most? Why?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$? Which function helps you find them?

What do you think the end behavior will be? Why?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when $g(x) = 0$? Do you think this will happen every time the denominator is equal to zero?

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At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

2. Let $r(x) = \dfrac{2x^2 + 7x - 4}{x^3 - 1}$.

What is the domain of $r(x)$?

What do you think the range of $r(x)$ will be?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

3. Let $r(x) = \dfrac{4x + 1}{4 - x}$.

What is the domain of $r(x)$?

What do you think the range of $r(x)$ will be?

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What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Now let's look at the graph of $r(x)$ using your calculator.

What value does $r(x)$ approach as x approaches infinity? Negative infinity? How could you describe this behavior? Why do you think this happens?

What occurs at the x-values when the denominator is equal to zero? Do you think this will happen every time?

At what x-values does $r(x)$ change signs (either + to – or vice-versa)? What else occurs at these x-values?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

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Now let’s summarize our findings and conclusions:

When is the domain of a rational function not $(-\infty, \infty)$? So what is your advice on how to determine the domain of a rational function?

Is the range of a rational function difficult to find? Why or why not?

How do you find the zeros or roots of a rational function?

How do you know where to find vertical asymptotes?

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

What possible things could occur at the x-values where a rational function changes signs?

Do you think it would be possible to use all of our knowledge of rational functions to create sketch without using the graphing calculator? Can you explain how this would work to another classmate?

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Graphing Rational Functions without a calculator…

Let $r(x) = \dfrac{3x^2 + 27}{2x^2 - 6x - 8}$.

What is the domain of $r(x)$?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

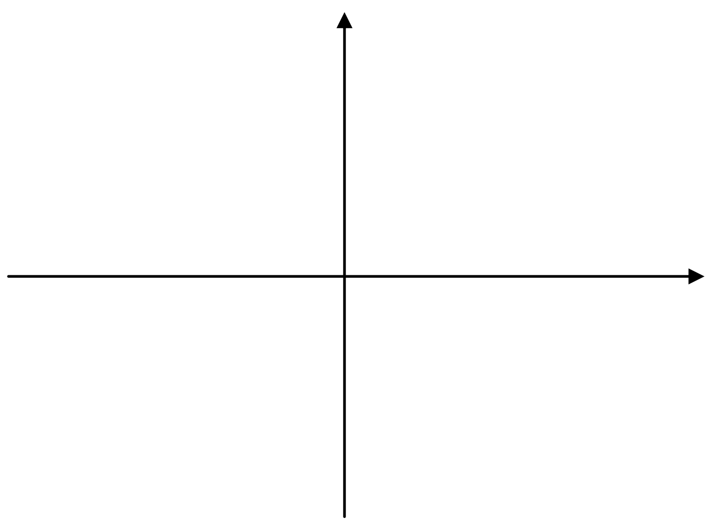
Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

Where is $r(x) > 0$? Where is $r(x) < 0$? (*Hint: use a sign chart)*

Now let's try to sketch the graph of $r(x)$ without using your calculator.



Based on your sketch, what do you think the range of $r(x)$ will be?

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

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Based on the graph from your calculator, what is the range of $r(x)$?

Try this one next. Let $r(x) = \dfrac{x^3 + 1}{3x^3 - 27x}$

What is the domain of $r(x)$?

What are the roots or zeros of $r(x)$?

What do you think the end behavior will be?

Where will $r(x)$ intersect the y-axis? How do you know?

Are there any vertical asymptotes? If so, where are they located?

Is there a horizontal asymptote? If so, where is it located?

At what x-values should $r(x)$ change signs (either + to – or vice-versa)? Why?

Where is $r(x) > 0$? Where is $r(x) < 0$?

Now let's try to sketch the graph of $r(x)$ without using your calculator.

Based on your sketch, what do you think the range of $r(x)$ will be?

Now let's compare your sketch to the graph of $r(x)$ using your calculator. How did your sketch match up with the actual curve? What characteristics of the graph were you not able to anticipate?

When is the function increasing? decreasing? Are there any local extremum? How can you be certain?

Based on the graph from your calculator, what is the range of $r(x)$?

In your own words, describe the process that you would go through in order to create a sketch of any rational function.

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Horizontal Asymptotes: How do we find them? Learning Task:

As we discuss the characteristics of rational functions, we know that it is important to consider the properties of the individual functions. Knowing about the individual functions helps us to know about the rational function. But as we discuss the details, let us consider the range values of the rational function. To do this it will be important consider the range values through a table and the graph. But to do this we are going to look at very large values for x.

1. Let $f(x)=5$ and $g(x)=x^2-6x+8$. Let $r(x)=\dfrac{f(x)}{g(x)}$. Complete the table of values with regards to the function given.

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the table in your graphing utility to get a better picture.

How does the graph relate to your table?

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2. Let $r(x)=\dfrac{2x^2+7x-4}{x^3-1}$. Complete the table of values with regards to the function given.

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the table in your graphing utility to get a better picture.
How does the graph relate to your table?

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3. Let $r(x) = \dfrac{4x+1}{4-x}$. Complete the table of values with regards to the function given.

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the table in your graphing utility to get a better picture.

How does the graph relate to your table?

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4. Let $r(x) = \dfrac{3x^2 + 27}{2x^2 - 6x - 8}$.

Complete the table of values with regards to the function given.

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

As the x-values head toward infinity, is there any significance to the y-values?

Examine the table in your graphing utility to get a better picture.

How does the graph relate to your table?

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5. Consider the range of the function, $R(x)=\dfrac{x^2-2x+5}{x-6}$. Complete the table of values with regards to the function given.

x	r(x)
1000	
700	
300	
200	
100	
0	
-100	
-200	
-300	
-700	
-1000	

What type of trends do you see from the y-values of this function?

.
As the x-values head toward infinity, is there any significance to the y-values?

Examine the table in your graphing utility to get a better picture.
How does the graph relate to your table?

Looking at the function, how is this one different from the others that we have considered?

Let's look back to Mathematics III and consider how long division can be helpful to us here.

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Now let's summarize our findings and conclusions:

When is the domain of a rational function not $(-\infty, \infty)$? So what is your advice on how to determine the domain of a rational function?

Is the range of a rational function difficult to find? Why or why not?

How do you find the zeros or roots of a rational function?

How do you know where to find vertical asymptotes?

What does a horizontal asymptote tell you about a rational function? Are they easy to locate? Do you know of any shortcuts to find them?

How do you know where a rational function will intersect the y-axis? Will a rational function always have a y-intercept? Can you give an example?

What possible things could occur at the x-values where a rational function changes signs?

Do you think it would be possible to use all of our knowledge of rational functions to create an accurate sketch without using the graphing calculator? Can you explain how this would work to another classmate?

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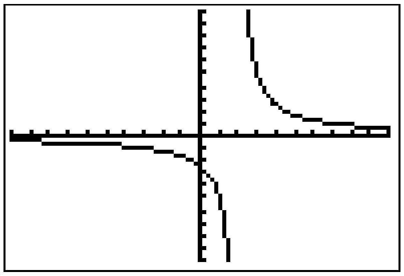
Inverse of a Rational Function Learning Task:

Let's consider the curve $f(x) = \dfrac{5}{x-2}$. Now let's look at some of the characteristics of $f(x)$.



What is the domain of $f(x)$?

What do you think the range of $f(x)$ will be?

What are the roots or zeros of $f(x)$?

What do you think the end behavior will be?

When will $f(x)$ intersect the y-axis?

Do you remember how to find the inverse of a curve algebraically? Please find $f^{-1}(x)$.

What is the domain of $f^{-1}(x)$?

What do you think the range of $f^{-1}(x)$ will be?

What are the roots or zeros of $f^{-1}(x)$?

What do you think the end behavior will be?

When will $f^{-1}(x)$ intersect the y-axis?

What are some special properties about inverses of functions that you remember?

Which of these properties will help us determine the range of $f(x)$?

Can we find the range of any rational function using this property of inverses? Why or why not?

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Let's try another example…

Let $r(x) = \dfrac{4x+1}{4-x}$. Find all the important characteristics of $r(x)$ and $r^{-1}(x)$.

For $r(x)$:

Domain:

Range:

Roots:

End Behavior:

y-intercept:

For $r^{-1}(x)$:

Domain:

Range:

Roots:

End Behavior:

y-intercept:

What important properties have you discovered between $r(x)$ and $r^{-1}(x)$? Is this true for all functions or just rational functions?

What is different about a function like $h(x) = \dfrac{4}{x^2 - 4x + 3}$ from the ones we have looked at so far?

Can you find $h^{-1}(x)$ algebraically? Why or why not?

If you cannot find $h^{-1}(x)$ algebraically, does that mean $h^{-1}(x)$ does not exist? Explain your reasoning. Would you always want to find the inverse of a rational function to determine its range?

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Partial Fraction Decomposition LearningTask:

Consider the two rational functions: $f(x)=\dfrac{3}{x+1}$ and $g(x)=\dfrac{2}{x-2}$. Using your prior knowledge of functions, graph both curves on the same axes.

Now suppose we wanted to investigate what happens with $f+g$. Express $f+g$ as one rational function $h(x)$.

What is the domain of $h(x)$?

What is the range of $h(x)$?

What are the roots or zeros of $h(x)$?

What do you think the end behavior will be?

Where will $h(x)$ intersect the y-axis?

Using the information from above, graph the rational function $h(x)$.

How does this graph compare with the first one you sketched? Describe the relationship between the two sketches.

What if we wanted to go the other direction? That is, what if we started off with one rational function and wanted to know the simpler rational functions that make up the one rational function? Is this possible? Let's try it…

Let $h(x)=\dfrac{4}{x^2-4x+3}$. Sketch a graph of $h(x)$.

Based on your sketch, how many simpler rational functions do you think make up $h(x)$? What would they be? Do you know for sure what their numerators would be?

Since we have no way of knowing exactly what the numerators should be, let's call them A and B for now. So we should have

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$$h(x) = \frac{4}{x^2 - 4x + 3} = \frac{A}{x-1} + \frac{B}{x-3}$$

How in the world can we figure out what A and B could possibly be? Maybe we could try guess and check? Does this seem like a good idea?

Since we are trying to compare one fraction to two different fractions, what might be a good idea now?

Since the denominators are equivalent, and the fractions are equal, we should be able to just compare the numerators. How is 4 going to equal an expression with 3 variables?

What should we do know?

Did that help you determine A and B? If not, what could we do next?

Does this help any? You know, something has really been bothering me in this problem. If these two expressions are supposed to be equal, why isn't there an x on the left side of the equal sign? Do you think we could put one in there just to make me feel better? What would the coefficient need to be in order to not change the problem?

If the two expressions have to be equal, then $A + B =$ ____? And $4 =$ __ If this is the case, can I now solve for A and for B? What does $A = ?$ and $B = ?$

So now we know that $h(x) = \dfrac{4}{x^2 - 4x + 3} = \dfrac{-2}{x-1} + \dfrac{2}{x-3}$. Do you see any similarities between our original curve we sketched and the two rational functions we just found?

As much fun as that was, I can already see new questions in your eyes: What if the denominator of our rational function doesn't factor into linear components? What if one of the factors is an irreducible quadratic? How is that situation going to be different from this one? Those are very good questions and definitely worth investigating.

Let's look at the rational function $g(x) = \dfrac{x^2 + 2x + 3}{x^3 - 1}$. When we factor the denominator, we see that we have one linear factor and one irreducible quadratic factor. That means, g(x) can be written as a sum of two other rational functions, one with a linear term in the denominator and one with a quadratic term in the denominator. Who knows what would go in the numerator? I'll let

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you in on a little secret… I'm not sure either, but I know how to find out. Let's use what we already know and put *A* over the linear term, so

$$g(x) = \frac{x^2 + 2x + 3}{x^3 - 1} = \frac{A}{x-1} + \frac{?}{x^2 + x + 1}.$$

It would be great if we could just put a *B* over the quadratic term, but something's just not right about that. You can try it if you like, but I don't think you will like the results. Since the denominator of the second fraction is a quadratic, I think I'm going to use a generic linear term in the numerator, something like $Bx + C$. Now I should have

$$g(x) = \frac{x^2 + 2x + 3}{x^3 - 1} = \frac{A}{x-1} + \frac{Bx + C}{x^2 + x + 1}.$$

From here, I should be able to duplicate the previous method of combining my two parts into one rational function and comparing the two numerators together. Let's see what happens.

Now you can find the values of *A*, *B*, and *C*. A = ___ B = ___ C = ___

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